Transport in unconventional superconductors: Application to liquid ^{3}He in aerogel

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We consider quite generally the transport of energy and momentum in unconventional superconductors and Fermi superfluids to which both impurity scattering (treated within the t-matrix approximation) and inelastic scattering contributes. A new interpolation scheme for the temperature dependence of the transport parameters is presented which preserves all analytical results available for $T \to 0$ and $T \to T_c$ and allows for a particularly transparent physical representation of the results. The two scattering processes are combined using Matthiessen's rule coupling. This procedure is applied for the first time to ^{3}He-B in aerogel. Here, at the lowest temperatures, a universal ratio of the thermal conductivity and the shear viscosity is found in the unitary limit, which is akin to the Wiedemann-Franz law.

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I. INTRODUCTION

Unconventional pairing correlations occur in a large class of Fermi systems including the superfluid phases of ^{3}He, heavy fermion superconductors, hole doped high-T_c cuprate superconductors, the Ruddlesden-Popper system Sr_2RuO_4, some organic superconductors and others. They are defined through the vanishing Fermi surface (FS) average of the energy gap $\langle \Delta_\mathbf{p} \rangle_{\mathrm{FS}} \equiv 0$ and manifest themselves by the sensitivity of thermodynamic, response and transport properties to the presence of even small concentrations of nonmagnetic impurities. For superfluid ^{3}He an elastic scattering mechanism in addition to the inelastic (two-particle) scattering is provided by porous silica aerogel (for a recent review see Ref. 1). Since the discovery of superfluidity of ^{3}He in aerogel[2] the analogy of this so-called dirty Fermi superfluid with dirty unconventional superconductors has been investigated in the literature.[3–12] An important qualitative effect of nonmagnetic impurities in unconventional superconductors is the occurrence of bound quasiparticle states in the gap which broaden into low energy impurity bands at finite concentrations of strong elastic scatterers.[13] It is this phenomenon which may give rise to interesting new consequences when applied to ^{3}He-B in aerogel. Motivated by the recent observation of an impurity-limited thermal conductivity[14] and viscosity[15] in ^{3}He-B we explore in this contribution for the first time both the impurity-limited transport in these systems for arbitrary scattering phase shifts [which includes both weak (Born) to strong (unitary) scattering] and effects of inelastic scattering processes. The latter are included via the Matthiessen rule approximation.

The paper is organized as follows: in Sec. II we describe the general concept of transport theory applicable to metallic (unconventional) superconductors and superfluid ^{3}He in aerogel. Elastic scattering processes are treated using the t-matrix approximation in the limit of s-wave scattering. To keep things on a simple level, an interpolation procedure for the temperature dependence of the transport parameters is proposed which preserves the exact asymptotic behavior near $T \to 0$ and $T \to T_c$. In Sec. III we apply this concept to the case of superfluid ^{3}He-B in aerogel. Here elastic (Sec. III A) and inelastic (Sec. III B) scattering is treated separately, the effects are eventually combined using the Matthiessen rule approximation. Sections IV and V are finally devoted to our discussion and conclusion.

II. UNIFIED TRANSPORT THEORY

A general transport parameter $\mathbf{T}^{aa}$ can be defined through the constitutive relation that connects a generalized current $\mathbf{j}^a$ with the gradient of its thermodynamically conjugate field δu^b,

$$j^a_\mu = - T^{ab}_{\mu\nu} \nabla_\nu \delta u^b,$$

$$(T^{aa}_{\mu\nu})_{e,i} = N_{\mathrm{F}} \left\langle 2 \int_0^\infty dE_\mathbf{p} \varphi_\mathbf{p} a^2_\mathbf{p} v_{\mathbf{p}\mu} v_{\mathbf{p}\nu} \tau_{e,i}(\widetilde{E}_\mathbf{p}) \right\rangle_{\mathrm{FS}},$$

$$\varphi_\mathbf{p} = - \frac{\partial \nu(E_\mathbf{p})}{\partial E_\mathbf{p}}, \quad \nu(E_\mathbf{p}) = \frac{1}{\exp(E_\mathbf{p}/k_B T) + 1}. \qquad (1)$$

Here $\tau_{e,i}$ denotes the transport time for elastic (e) and inelastic (i) scattering, which differs, in general, from the quasiparticle relaxation time due to the existence of vertex corrections. In (1) $a_\mathbf{p}$ is a general vertex which classifies the transport process of interest. Thus for the diffusive thermal conductivity, $\mathbf{j}^a$ is the entropy current, and its associated vertex $a_\mathbf{p}$ is the energy dispersion $E_\mathbf{p}$ of the Bogoliubov quasiparticles (bogolons),

$$a_\mathbf{p} = E_\mathbf{p} = \sqrt{\xi^2_\mathbf{p} + \Delta^2_\mathbf{p}},$$

with $\Delta_\mathbf{p}$ the energy gap, $\xi_\mathbf{p} = \epsilon_\mathbf{p} - \mu$ the normal state energy measured from the chemical potential μ and δu^b represents the temperature change $\delta T/T$. For the shear viscosity, $\mathbf{j}^a$ is the momentum current, the vertex reads

$$a_\mathbf{p} = \mathbf{p},$$

and δu^b represents the normal velocity field $\mathbf{v}^{\mathrm{n}}$. In Eq. (1) N_{F} denotes the density of states for both spin projections, $\langle \cdots \rangle_{\mathrm{FS}}$

denotes the Fermi surface (angular) average, and the index $e(i)$ refers to elastic (inelastic) scattering.

A. Impurity-limited transport

The impurity-limited transport time $\tau_e(\tilde{E}_{\mathbf{p}})$ is defined by[17]

$$\tau_e(\tilde{E}_{\mathbf{p}}) = \frac{\hbar}{\mathrm{Im}\,\sqrt{\tilde{E}_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2}}\frac{1}{2}\left(1 + \frac{|\tilde{E}_{\mathbf{p}}^2| - \Delta_{\mathbf{p}}^2}{|\tilde{E}_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2|}\right). \quad (2)$$

Here $\tilde{E}_{\mathbf{p}}$ is the bogolon energy renormalized by the impurity self-energy (computed within the t-matrix approximation in the limit of s-wave scattering[17]),

$$\tilde{E}_{\mathbf{p}} = E_{\mathbf{p}} + \Sigma_e(\tilde{E}_{\mathbf{p}}),$$

$$\Sigma_e(\tilde{E}_{\mathbf{p}}) = i\Gamma_N^e \frac{D(\tilde{E}_{\mathbf{p}})(1+c^2)}{c^2 + D^2(\tilde{E}_{\mathbf{p}})},$$

$$D(\tilde{E}_{\mathbf{p}}) = \left\langle \frac{\tilde{E}_{\mathbf{p}}}{\sqrt{\tilde{E}_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2}} \right\rangle_{\mathrm{FS}}. \quad (3)$$

In Eq. (3)

$$\Gamma_N^e = \frac{1}{\tau_N^e} = \frac{2n_i}{\pi N_{\mathrm{F}}}\frac{1}{(1+c^2)}$$

represents the normal state scattering rate, with n_i the impurity concentration. The parameter $c = \cot\,\delta_0$ is related to the s-wave scattering phase shift δ_0. It is worth emphasizing that the use of the impurity t-matrix approximation has been crucial for the understanding of the temperature dependence of the transport parameters in heavy Fermion superconductors, which strongly disagreed with calculations performed in the Born scattering limit.[18] Note that the superconducting density of states is related to $D(\tilde{E}_{\mathbf{p}})$ by

$$\frac{N_s(\tilde{E}_{\mathbf{p}})}{N_0} = \mathrm{Re}\,D(\tilde{E}_{\mathbf{p}}).$$

The impurity self-energy has the following low energy behavior:

$$\Sigma_e(\tilde{E}_{\mathbf{p}}) = \Sigma_e'(\tilde{E}_{\mathbf{p}}) + i\Sigma_e''(\tilde{E}_{\mathbf{p}}),$$

$$\Sigma_e'(\tilde{E}_{\mathbf{p}}) \overset{E_{\mathbf{p}}\to 0}{=} 0,$$

$$\Sigma_e''(\tilde{E}_{\mathbf{p}}) \overset{E_{\mathbf{p}}\to 0}{=} \Sigma_e''(0). \quad (4)$$

$\tau_e(\tilde{E}_{\mathbf{p}})$ can be evaluated analytically in the limits $E_{\mathbf{p}} \ll \Delta_0$ and $E_{\mathbf{p}} \gg \Delta_0$ (with Δ_0 the gap maximum):

$$\tau_e(\tilde{E}_{\mathbf{p}}) = \begin{cases} \dfrac{\hbar\Sigma_e''^2(0)}{[\Delta_{\mathbf{p}}^2 + \Sigma_e''^2(0)]^{3/2}}, \quad \tilde{E}_{\mathbf{p}} \ll \Delta_0, \\ \tau_N^e \dfrac{E_{\mathbf{p}}}{\sqrt{E_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2}} \dfrac{1 + c^2[D(E_{\mathbf{p}})]^{-2}}{(1+c^2)D(E_{\mathbf{p}})}\Theta(E_{\mathbf{p}} - \Delta_{\mathbf{p}}), \\ \tilde{E}_{\mathbf{p}} \gg \Delta_0, \quad \hbar\Gamma_N^e \ll \Delta_0. \end{cases} \quad (5)$$

In (5) Θ denotes the Heaviside step function. Note that due to the assumption of s-wave scattering, vertex corrections to the elastic part of the transport coefficients do not occur. Equation (5) can be used to calculate both the zero temperature limit and the temperature dependence of $T_{\mu\nu}^{aa}$. In the zero temperature limit we may define a dimensionless quantity $C_{\mu\nu}^{aa}$ through

$$C_{\mu\nu}^{aa} = \frac{\hbar\Gamma_N^e}{\langle a^2(\hat{\mathbf{p}})\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu\rangle_{\mathrm{FS}}}\left\langle \frac{\Sigma_e''^2(0)a^2(\hat{\mathbf{p}})\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu}{[\Sigma_e''^2(0) + \Delta_{\mathbf{p}}^2]^{3/2}} \right\rangle_{\mathrm{FS}}. \quad (6)$$

Here $a(\hat{\mathbf{p}})$ denotes the $\hat{\mathbf{p}}$-dependent part of the vertex $a_{\mathbf{p}}$. The tensor $C_{\mu\nu}^{aa}$ describes a normal-like low T contribution to the quasiparticle transport induced by resonant pair-breaking as a consequence of strong impurity scattering in the unitary limit. The pair breaking parameter $C_{\mu\nu}^{aa}$ drops rapidly with decreasing δ_0 and vanishes for a critical value δ_{0c} which depends on the gap symmetry. Note that in the unitary limit $C_{\mu\nu}^{aa}/\Gamma_N^e$ turns out to be independent of the parameter Γ_N^e under certain conditions. This phenomenon is known as universal transport in unconventional (nodal) superconductors. It was first proposed by Lee[19] and has been observed in various experiments.

B. Interpolation procedure for $(\mathbf{T}^{aa})_e$

The temperature dependence of $T_{\mu\nu}^{aa}$ can be estimated by observing that the self-consistent solution of Eq. (3) is relevant only for low energies. At high energies one may use the asymptotic form of $\tau_e(\tilde{E}_{\mathbf{p}})$ for $E_{\mathbf{p}} \gg \Delta_0$. This gives rise to the definition of T-dependent generalized Yosida functions,

$$Y_{\mu\nu}^{aa(n)}(T) = \frac{\left\langle 2\displaystyle\int_{\Delta_{\mathbf{p}}}^{\infty} \frac{dE_{\mathbf{p}}E_{\mathbf{p}}}{\sqrt{E_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2}}\varphi_{\mathbf{p}}\frac{a_{\mathbf{p}}^2\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu}{|D(E_{\mathbf{p}})|^n} \right\rangle_{\mathrm{FS}}}{\left\langle \displaystyle\int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\varphi_{\mathbf{p}}^N a_{\mathbf{p}}^2\hat{\mathbf{p}}_\mu\hat{\mathbf{p}}_\nu \right\rangle_{\mathrm{FS}}}. \quad (7)$$

Here

$$\varphi_{\mathbf{p}}^N = \lim_{\Delta\to 0}\varphi_{\mathbf{p}}.$$

From (6) and (7) one may construct the following interpolation procedure for the transport coefficient $T_{\mu\nu}^{aa}(T)$:

$$\frac{(T_{\mu\nu}^{aa})_e}{(T_{\mu\nu}^{aa})_N^e} = C_{\mu\nu}^{aa} + (1 - C_{\mu\nu}^{aa})\frac{Y_{\mu\nu}^{aa(1)} + c^2Y_{\mu\nu}^{aa(3)}}{1+c^2}. \quad (8)$$

Equation (8) is one central result of this paper. It describes the temperature dependence of the generalized transport parameter $(T_{\mu\nu}^{aa})_e$ as related to its normal state counterpart for arbitrary scattering phase shifts $c=\cot\delta_0$. Only for a certain range of s-wave scattering phase shifts δ_0, not too far below the unitary limit value $\pi/2$, the offsets $C^{aa}_{\mu\nu}$ are finite and lead to a contribution from normal quasiparticles which originate from resonant pair breaking by the impurities. The temperature dependence of $T^{aa}_{\mu\nu}$ is then described by the system of generalized Yosida functions $Y^{aa(n)}_{\mu\nu}(T)$. The interpolation (8) has the following virtues. (i) It is exact in the limit $T\to 0$ (for arbitrary values of the impurity scattering phase shifts δ_0) and just below the transition temperature (if $\hbar\Gamma^e_N \ll \Delta$). (ii) It is, though approximate, a physically transparent representation of the behavior of $T^{aa}_{\mu\nu}$ at intermediate temperatures in terms of the generalized Yosida functions $Y^{\mu\nu(n)}_{ab}(T)$. The latter can even be evaluated analytically.[23] (iii) The treatment is valid for a general vertex $a_{\mathbf{p}}$ and hence unifies the description of the shear viscosity and the (diffusive) thermal conductivity.

C. Inelastic scattering

Finally, inelastic scattering is accounted for by invoking the Matthiessen rule approximation,

$$\frac{1}{T^{aa}_{\mu\nu}} = \left(\frac{1}{T^{aa}_{\mu\nu}}\right)_e + \left(\frac{1}{T^{aa}_{\mu\nu}}\right)_i . \tag{9}$$

The inelastic contributions to $T^{\mu\nu}_{aa}$, which may originate, for example, from phonon or spin fluctuation scattering, must be discussed for every system separately.

III. APPLICATION TO ^{3}He-B IN AEROGEL

In what follows, we wish to apply the theory to ^{3}He-B with silica aerogel acting as an impurity system. The latter is treated as a homogeneous isotropic scattering medium (HISM).[8] Such an assumption is valid when the average spacing between the aerogel strands is smaller than the superfluid coherence length, and is fulfilled at low pressure and not too low aerogel concentrations. It is well established that the ^{3}He-aerogel system can be described by a spin triplet p-wave order parameter of the form (BW state)

$$\Delta^2_{\mathbf{p}} = \mathbf{d}_{\mathbf{p}} \cdot \mathbf{d}^*_{\mathbf{p}} = \Delta^2$$

with the vector $\mathbf{d}_{\mathbf{p}}$ characterizing the three triplet components.

A. Elastic scattering

We begin by estimating the elastic quasiparticle mean free path λ_a by the geometric mean free path λ_g in correlated aerogel,[1,9]

$$\lambda_a \approx \lambda_g = \frac{0.279\times 10^{-4}}{c_a^{1.1}}\ \text{cm},$$

where c_a represents the aerogel concentration in %. The elastic scattering rate is obtained as $\Gamma^e_N = v_F/\lambda_a$, with v_F the Fermi velocity. For elastic scattering

$$D(\tilde{E}_{\mathbf{p}}) = \frac{\tilde{E}_{\mathbf{p}}}{\sqrt{\tilde{E}^2_{\mathbf{p}} - \Delta^2}}$$

with Δ the energy gap in the presence of impurities. The zero energy limit of the impurity self-energy can be evaluated analytically for arbitrary impurity scattering phase shifts,

$$\Sigma''_0(0) = \left[\left(\frac{\Gamma^{e2}_N\Delta^2}{1+c^2} + \frac{\Gamma^{e4}_N}{4}\right)^{1/2} + \frac{\Gamma^{e2}_N}{2} - \frac{\Delta^2 c^2}{1+c^2}\right]^{1/2}, \tag{10}$$

$\Sigma''_0(0)$ is seen to decrease with increasing $c=\cot\delta_0$ and to vanish for $c=[\Gamma^e_N/(\Delta-\Gamma^e_N)]^{1/2}=(2n_i/\pi N_F\Delta)^{1/2}$. Due to the isotropy of Δ, the transport tensors simplify according to

$$(\kappa_{\mu\nu})_e = \kappa_e \delta_{\mu\nu},$$

$$(\eta^{\mu\nu}_{ij})_e = \eta_e \delta_{\mu\nu}\delta_{ij}.$$

The result for the shear viscosity reads

$$\eta_e(T) = \eta^e_N t^e_\eta(T),$$

$$t^e_\eta(T) = C_\eta + (1-C_\eta)\frac{Y^\eta_1(T) + c^2 Y^\eta_3(T)}{1+c^2},$$

$$C_\eta = \frac{\hbar\Gamma_N \Sigma''^2_0(0)}{[\Sigma''^2_0(0)+\Delta^2]^{3/2}},$$

$$Y^\eta_n = \int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\varphi_{\mathbf{p}} \left|\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right|^n . \tag{11}$$

Here

$$\eta^e_N = \tfrac{1}{5} n p_F v_F \tau^e_N$$

is the impurity-limited shear viscosity of the normal state. In (11) the variable $\xi_{\mathbf{p}} = \epsilon_{\mathbf{p}} - \mu$ measures the energy $\epsilon_{\mathbf{p}}$ from the chemical potential μ. The result for the thermal conductivity reads

$$\kappa_e(T) = \kappa^e_N t^e_\kappa(T),$$

$$t^e_\kappa(T) = C_\kappa + (1-C_\kappa)\frac{Y^\kappa_1(T) + c^2 Y^\kappa_3(T)}{1+c^2},$$

$$C_\kappa \equiv C_\eta,$$

$$Y^\kappa_n = \int_{-\infty}^{\infty} d\xi_{\mathbf{p}}\varphi_{\mathbf{p}} \left|\frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}}\right|^n \left(\frac{E_{\mathbf{p}}}{2k_BT}\right)^2 . \tag{12}$$

Here

$$T\kappa^e_N = \frac{1}{3}\frac{n}{m^*}(\pi k_B T)^2 \tau^e_N$$

is the impurity-limited thermal conductivity of the normal state. We note that our result (12) is in accord with a previous calculation of Sharma and Sauls.[10] In Fig. 1 we show the reduction of the pair-breaking parameter C_η as the s-wave scattering phase shift δ_0 decreases from its unitary limit

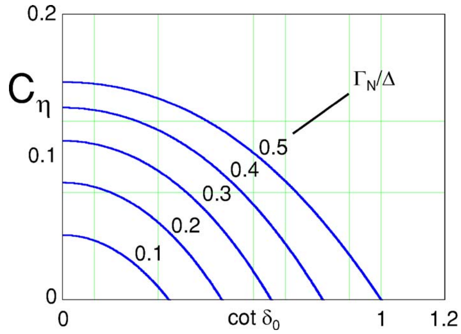

FIG. 1. (Color online) Pair-breaking parameter C_η for ^{3}He-B for various Γ_N/Δ vs scattering phase shift $c=\cot\delta_0$.

value $\pi/2$. We emphasize that in contrast to nodal superconductors like hole-doped cuprates, the quantities C_κ/Γ_N^e and C_η/Γ_N^e are nonuniversal, i.e., they depend on the parameters Γ_N^e and δ_0. It is interesting to study the ratio of the two transport parameters in the unitary limit $c\to 0$,

$$\frac{\kappa_e(T)}{\eta_e(T)T}=\frac{5}{3}\left(\frac{\pi k_B}{p_F}\right)^2\frac{t_\kappa^e(T)}{t_\eta^e(T)}\overset{T\to 0}{=}\frac{5}{3}\left(\frac{\pi k_B}{p_F}\right)^2. \tag{13}$$

This means that although C_κ/Γ_N^e and C_η/Γ_N^e are nonuniversal, the ratio of the impurity-limited diffusive thermal conductivity κ_e and the shear viscosity η_e of superfluid ^{3}He-B tends to a universal value independent of Γ_N^e in the unitary limit $c=0$ at $T=0$. This is somehow similar to the Wiedemann-Franz law that connects the thermal and electronic conductivities in nodal unconventional superconductors at $T=0$ in the unitary limit.[20]

B. Inelastic scattering

The relaxation rate for inelastic scattering can be written in the form[21]

$$\frac{1}{\tau_i(E_\mathbf{p})}=\begin{cases}\dfrac{1}{\tau_N^i(T)}\left[1+\left(\dfrac{\xi_\mathbf{p}}{\pi k_B T}\right)^2\right], & T>T_c,\\[2ex] \dfrac{E_\mathbf{p}}{\sqrt{E_\mathbf{p}^2-\Delta^2}}\dfrac{1}{\tau_B(E_\mathbf{p})}, & T<T_c.\end{cases} \tag{14}$$

Here

$$\tau_N^i(T)=\tau_N^i(T_c)\left(\frac{T_c}{T}\right)^2$$

is the inelastic relaxation time of the normal Fermi liquid. The relaxation time $\tau_B(E_\mathbf{p})$ has been discussed in Ref. 21. At low temperatures it has the asymptotic form

$$\lim_{T\to 0}\frac{1}{\tau_B(E_\mathbf{p})}=\frac{1}{\tau_B^0}=\frac{3w_0 e^{-\Delta/k_B T}}{\sqrt{2\pi}\tau_N^i}\left(\frac{\Delta}{k_B T}\right)^{3/2} \tag{15}$$

with $w_0\approx 1$. For our purposes it is sufficient to work with the thermal average,

TABLE I. Relevant Fermi liquid parameters for ^{3}He.

P(bar)	v_F(cm/s) (Ref. 24)	$\bar{\tau}_N^i(T_c)(10^{-8}$ s) (Ref. 25)
0	5890	37.50
10	4650	9.00
20	3810	4.05
30	3360	3.00

$$\frac{1}{\bar{\tau}_B(T)}=\frac{\sum_{\mathbf{p}\sigma}\varphi_\mathbf{p}/\tau_B(E_\mathbf{p})}{\sum_{\mathbf{p}\sigma}\varphi_\mathbf{p}}=\begin{cases}\dfrac{1}{\bar{\tau}_N^i(T)}\equiv\dfrac{4}{3}\dfrac{1}{\tau_N^i(T)}, & T>T_c,\\[2ex] \dfrac{1}{\tau_B^0(T)}, & T<T_c.\end{cases}$$

One may now interpolate the full temperature-dependent thermally averaged bogolon relaxation rate along the lines of the procedure proposed in Ref. 23 to get

$$\frac{1}{\bar{\tau}_B(T)}=\frac{Y_0^\eta(T)}{\bar{\tau}_N^i(T)}\left(\frac{9w_0\Delta(0)}{8\pi k_B T}(1-t^\alpha)+t^{\alpha-1}\right) \tag{16}$$

with $t=T/T_c$. The exponent α must be chosen such that the slope of the interpolated bogolon relaxation rate coincides with that of an exact numerical evaluation.[21]

In Table I we list two pressure-dependent parameters for the Fermi liquid ^{3}He, that enter into the calculation of inelastic scattering effects.

Finally the inelastic contribution to the transport parameters η_i and κ_i, which include the appropriate vertex corrections, can be approximated as

$$\eta_i=\eta_N^i t_\eta^i,$$

$$\kappa_i=\kappa_N^i t_\kappa^i,$$

$$t_{\eta,\kappa}^i=Y_2^{\eta,\kappa}\frac{\bar{\tau}_B}{\bar{\tau}_N^i}\frac{1-\lambda_{\eta,\kappa}}{1-\lambda_{\eta,\kappa}\dfrac{Y_2^{\eta,\kappa}}{Y_0^{\eta,\kappa}}}, \tag{17}$$

where $Y_n^{\eta,\kappa}$ is defined through Eqs. (12) and (13) and $\lambda_{\eta,\kappa}$ are pressure-dependent scattering parameters that account for the vertex corrections in the collision integral.[21] It should be emphasized that experimental results on the shear viscosity of clean superfluid ^{3}He-B could be quantitatively explained by Eq. (17) after invoking corrections due to the proximity of the Knudsen transition.[22] Note that in the absence of mean free path corrections, the inelastic transport parameters $\kappa_i T$ and η_i are seen to have finite low-T values, as seen from Eq. (17),

$$\lim_{T\to 0}\kappa_i T=\frac{n}{m^*}\Delta^2\tau_s,$$

$$\lim_{T\to 0}\eta_i=\tfrac{1}{5}np_F v_F\tau_s, \tag{18}$$

in which

TABLE II. λ_a, c_a, and γ_N at a pressure of 10 bar for various aerogel concentrations c_a.

$c_a(\%)$	λ_a(nm)	a_c	γ_N
0.1	3511	1.19	0.032
0.2	1638	2.55	0.069
1.0	279	15.00	0.408
2.0	130	32.15	0.874

$$\tau_s = \frac{8\pi\bar{\tau}_N^i}{9w_0}\left(\frac{k_BT}{\Delta}\right)^2$$

with a ratio

$$\lim_{T\to 0}\frac{\kappa_i T}{\eta_i} = 5\frac{\Delta^2}{p_F^2}.$$

As a consequence of (9), the total shear viscosity can be written in the unitary limit ($c=0$) at $T\to 0$ as

$$\lim_{T\to 0}\eta(T) = \frac{1}{5}\frac{np_Fv_F\tau_N^e C_\eta}{1+\dfrac{\tau_N^e}{\tau_s}C_\eta}. \tag{19}$$

Clearly, the Matthiessen rule leads to a superposition of two constant low-T limits of the shear viscosities η_e and η_i. In contrast, the total diffusive thermal conductivity has a nontrivial temperature dependence in the unitary limit near $T\to 0$,

$$\lim_{T\to 0}\kappa(T) = \frac{n}{m^*}\frac{\pi^2k_B^2T\tau_N^e C_\kappa}{3+\dfrac{\tau_N^e C_\kappa}{\tau_s}\left(\dfrac{\pi k_B T}{\Delta}\right)^2} \tag{20}$$

and therefore behaves very differently depending on whether impurity scattering dominates, in which case $\kappa(T)$ varies $\propto T$ to varying $\propto T^{-1}$ if inelastic scattering dominates.

IV. DISCUSSION

The numerical analysis of the expressions for the total shear viscosity η and thermal conductivity κ can be characterized by two parameters that depend on aerogel concentration and pressure. The first

$$a_c = \frac{\bar{\tau}_N^i(T_c)}{\tau_N^e}$$

describes the relative importance of elastic and/or inelastic scattering and one has $a_c\to 0$ in the clean limit. The second

$$\gamma_N = \lim_{c\to 0}\frac{\hbar\Gamma_N^e}{\Delta}$$

parametrizes the strength of the $T=0$ offset in η and κ/T caused by resonant pair breaking caused by the impurity system in the unitary limit illustrated in Figs. 2 and 3.

In Table II we list values of the elastic mean free path λ_a as well as the parameters a_c and γ_N evaluated in the range of experimentally relevant aerogel concentration, c_a at a pressure of 10 bar.

It should be noted that the pressure dependence of the results for the total shear viscosity η and thermal conductivity κ enters mainly through the parameters a_c and γ_N.

In Figs. 2 and 4 we show numerical evaluations of the total reduced viscosity according to (9) for unitary and Born scattering, respectively, at a pressure of 10 bar. In the unitary case, for not too large values of the parameter $a_c = \tau_N^i(T_c)/\tau_N^e$, the viscosity is seen to almost vanish in the zero temperature limit. If, on the other hand, $a_c=10$ (correspond-

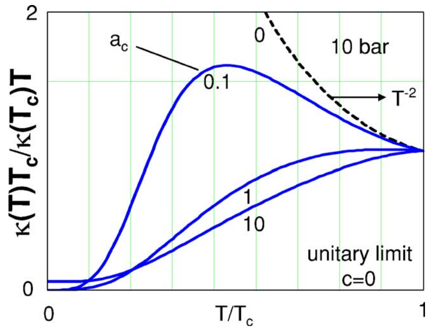

FIG. 3. (Color online) Total reduced thermal conductivity, divided by T/T_c, at a pressure of 10 bar in the unitary limit vs reduced temperature for various values of the parameter $a_c = \bar{\tau}_N^i(T_c)/\tau_N^e$. Again, $a_c=0$ corresponds to the clean limit.

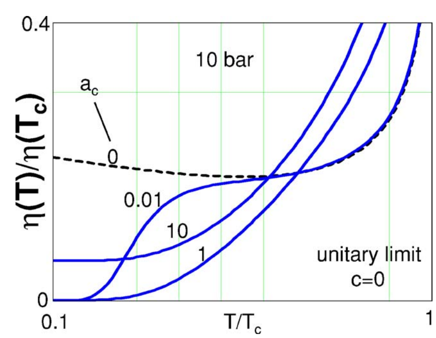

FIG. 2. (Color online) Total reduced shear viscosity at a pressure of 10 bar in the unitary limit vs reduced temperature for various values of the parameter $a_c = \bar{\tau}_N^i(T_c)/\tau_N^e$. Note that $a_c=0$ corresponds to the clean limit.

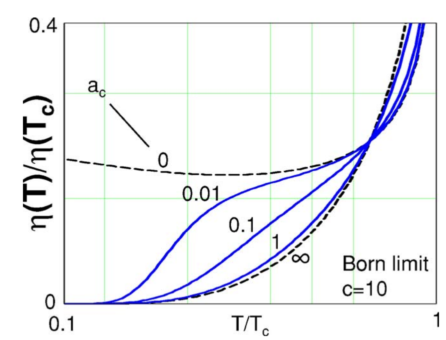

FIG. 4. (Color online) Total reduced shear viscosity in the Born limit vs reduced temperature for various values of the parameter $a_c = \bar{\tau}_N^i(T_c)/\tau_N^e$.

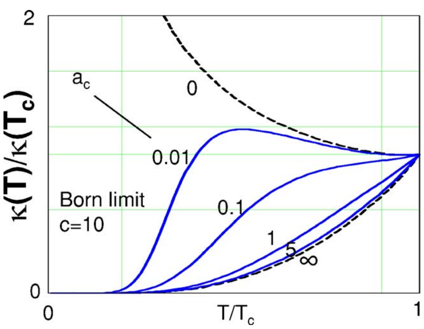

FIG. 5. (Color online) Total reduced thermal conductivity in the Born limit vs reduced temperature for various values of the parameter $a_c = \bar{\tau}_N^i(T_c)/\tau_N^e$.

ing to approximately 99.3% open aerogel) the viscosity has a finite low-T limit. For Born scattering (Fig. 4) the viscosity falls off much more rapidly ($\propto Y_3^\eta$) at low T. It is interesting to note that crossing points like those observable in Fig. 4 (Born limit) for the shear viscosity vs reduced temperature, which generally occur in context with a superposition law,[16] arise here as a consequence of the Matthiessen rule approximation.

According to (20) the thermal conductivity behaves differently $\kappa(T) \propto T(\propto T^{-1})$ if impurity (inelastic) scattering dominates, respectively. The reduced total thermal conductivity [Eq. (9)], is shown for a pressure of 10 bar in Fig. 3 [here divided by (T/T_c)] and Fig. 5 for unitary and Born scattering, respectively. For $a_c = 10$ (corresponding to approximately 99.3% open aerogel) the offset in κ/T can clearly be seen and corresponds to a linear dependence of $\kappa(T)$ on T, predicted for unitary scattering. The latter behavior is in qualitative agreement with the experimental results at Lancaster.[14] For Born scattering (Fig. 5) the thermal conductivity falls off much more rapidly ($\propto Y_3^\kappa$) at low T. The ratio of the transport parameters κ/T and η in the unitary limit at $T \to 0$ is seen to be universal only in the impurity-limited unitary case $\tau_N^i \gg \tau_N^e$. The latter condition is fulfilled for superfluid He in aerogel for a large range of aerogel concentrations.

V. SUMMARY AND CONCLUSION

In summary, we provide a comprehensive picture of the transport of momentum and energy in unconventional superconductors, here with particular emphasis on superfluid ^{3}He-B in aerogel. Strong elastic scattering is treated quite generally (using the impurity t-matrix) for both superconductors and dirty Fermi superfluids. A new interpolation scheme for the temperature dependence of the transport parameters is presented, which on the one hand preserves all analytical results available for $T \to 0$ and $T \to T_c$ and allows for a particularly transparent physical representation of the transport parameters which would otherwise be quite complex in structure, given the diversity of effects that enter the general description of transport. New physics is seen to arise in the unitary limit (e.g, scattering phase shift $\delta_0 = \pi/2$) and the phenomenon of universal transport occurs for superconductors with gap nodes. For the ^{3}He-B/aerogel system both the impurity-limited values of $\kappa/T, \eta$ are proportional to the same nonuniversal constant C_η, and interestingly give rise to a universal relation in a manner akin to the Wiedemann-Franz law. We show that inelastic scattering can be accounted for quite accurately in the case of superfluid ^{3}He-B and leads to the well-known low-T asymptotic behavior of $\kappa T, \eta \to$ const in the clean limit. In this contribution, we have, for the first time, combined both effects to yield a comprehensive picture of the transport properties of the ^{3}He-B/aerogel system. These are seen to be impurity limited for a wide range of experimentally accessible aerogel concentrations. A comparison of theory with future experiments may discriminate whether the impurity-limited transport is characterized by scattering in the Born or the unitary limit and to determine the nature of change in δ_0 with the addition of ^{3}He.[15] The dominance of the unitary limit is supported, for example, by Refs. 14 and 15. The theoretical concept presented in this contribution clearly demonstrates strong parallels in the transport properties of strikingly different systems: dirty unconventional metallic superconductors and the superfluid phases of ^{3}He in aerogel.

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