PHYSICAL REVIEW B, VOLUME 63, 140507(R)

Sound conversion in impure superfluids

Peter Brusov, J. M. Parpia, Paul Brusov, and G. Lawes
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York, 14853

(Received 18 September 2000; published 21 March 2001)

We consider the conversion of first sound into second sound and the reverse in different impure superfluid systems. We calculate the coupling between temperature (entropy) oscillations and pressure (density) oscillations for impure superfluids (including ^{3}He-^{4}He mixtures) and for superfluid He (both ^{3}He and ^{4}He) in aerogel and show that sound conversion in these systems is governed by $c(\partial\rho/\partial c)$ or by $\sigma\rho_a\rho_s$ (instead of $\partial\rho/\partial T$, which is enormously small in pure He). This replacement plays a fundamental role in the sound conversion phenomenon in impure superfluids and superfluids in aerogel. It enhances the coupling strengths of the two sounds and decreases the threshold values for nonlinear processes in homogeneous superfluids. Some phenomena such as the slow mode of superfluid ^{3}He and ^{4}He in aerogel, and heat pulse propagation with the velocity of first sound in HeII in aerogel can be understood as a double sound conversion phenomena.

DOI: 10.1103/PhysRevB.63.140507 PACS number(s): 67.40.Pm, 67.57.Pq

A bulk superfluid supports two sound modes; first sound which is ordinary sound propagated with a velocity $u_1 = \sqrt{(\partial P/\partial\rho)_\sigma}$ corresponding to pressure (density) oscillations at constant entropy density and second sound, that propagates with a velocity $u_2 = \sqrt{[\sigma^2\rho_s/\rho_n(\partial\sigma/\partial T)]}$ representing temperature (entropy) oscillations. The possibility of propagating undamped temperature waves is unique to superfluids. In the normal state, thermal waves cannot propagate because they are damped over a distance of order the wavelength, $\lambda \approx \sqrt{\chi/\omega}$ through conductive heat transfer (χ is the thermal conductivity coefficient).[1]

These sound modes are coupled through the thermal expansion coefficient $\partial\rho/\partial T$, which is small in all fluids and is anomalously small for superfluid HeII. Sound conversion phenomena have been predicted by Khalatnikov for two strongly inhomogeneous cases: reflection of second sound from the boundary between HeII and its vapor[1] and sound reflection from the boundary between the ^{3}He-rich phase and the ^{4}He-rich phase of a ^{3}He-^{4}He mixture.[2]

Nonlinear sound conversion phenomena for homogeneous pure superfluids have been considered by a few authors.[3,4] Parametric decay of first sound[5] and conversion of two second sounds into first sound[6] in pure superfluids have been observed experimentally.

In this paper we consider sound conversion phenomena for impure homogeneous superfluids, where these phenomena are caused either by impurities (including ^{3}He in HeII) or by the presence of aerogel. We refer to superfluid He in aerogel as a homogeneous superfluid in order to distinguish this case from the strongly inhomogeneous cases considered by Khalatnikov. In the cases that we consider, the coupling between the two sound modes is provided by $c(\partial\rho/\partial c)$ (c is the impurity concentration) or by $\sigma\rho_s\rho_a$ (ρ_a is the aerogel density) (instead of $\partial\rho/\partial T$ which is vanishingly small in pure HeII). This coupling found by us in linear approximation plays an important role in sound conversion phenomena. Even the existence of a vanishing coupling $\partial\rho/\partial T$ in pure superfluids changes significantly all possible nonlinear processes: Cerenkov emission, decay of first sound to second sound, and conversion of second sound into first sound. Cerenkov emission is absent at $\partial\rho/\partial T = 0$, while in the parametric decay process terms $\propto \partial\rho/\partial T$ are the same order as the leading terms.[3] Our calculations show that coupling of sounds in impure superfluids is much larger than in pure superfluids. This increases strongly the possibility of observing sound conversion phenomena.

We start by showing the decoupling of first and second sound in pure superfluids. This section also illustrates the general framework of our calculations. In the following section we consider sound conversion in impure superfluids and show that this process is governed by a term proportional to $c(\partial\rho/\partial c)$ instead of $\partial\rho/\partial T$ as in pure superfluids. We then calculate the coupling between the two sound modes for superfluid He in aerogel where the interaction is provided by a term proportional to $\sigma\rho_a\rho_s$.

We next consider the double sound conversion (DSC) phenomenon in which first sound is converted into second sound which is then converted back into first sound, and the inverse. It is possible that DSC can explain previous experimental results such as the observation of the slow mode in superfluid ^{3}He and HeII in aerogel[7,8] and the propagation of a heat pulse with the velocity of the fast mode in HeII in aerogel[9] which were not understood at that time. In the conclusion we summarize our results.

I. CASE OF PURE SUPERFLUIDS

Sound propagation in a superfluid is described by the following equations obtained by linearizing the two fluid hydrodynamical equations:[1]

$$\frac{\partial^2\rho}{\partial t^2} = \Delta P; \quad \frac{\partial^2\sigma}{\partial t^2} = \frac{\rho_s}{\rho_n}\sigma^2\Delta T, \tag{1}$$

where $\sigma = s/\rho$, s is the entropy density and $\rho = \rho_s + \rho_n$ is the total mass density. After substituting $P = P_0 + P'$ and $T = T_0 + T'$ (where f_0 is the equilibrium value and f' is the perturbation due to the sound wave) we obtain:

$$\frac{\partial\rho}{\partial P}\frac{\partial^2 P'}{\partial t^2} - \Delta P' + \frac{\partial\rho}{\partial T}\frac{\partial^2 T'}{\partial t^2} = 0 \tag{2}$$

$$\frac{\partial\sigma}{\partial P}\frac{\partial^2 P'}{\partial t^2} + \frac{\partial\sigma}{\partial T}\frac{\partial^2 T'}{\partial t^2} - \frac{\sigma^2\rho_s}{\rho_n}\Delta T' = 0. \tag{3}$$

If we consider a plane wave, P' and T' are proportional to $\exp[-i\omega(t-x/u)]$. We can rewrite Eqs. (2) and (3) as:

$$\left(\frac{\partial\rho}{\partial P}u^2-1\right)P'+\frac{\partial\rho}{\partial T}u^2T'=0 \tag{4}$$

$$\frac{\partial\sigma}{\partial P}u^2P'+\left(\frac{\partial\sigma}{\partial T}u^2-\frac{\sigma^2\rho_s}{\rho_n}\right)T'=0. \tag{5}$$

The secular equation of this system gives us two velocities for sound propagation: $u_1=\sqrt{(\partial P/\partial\rho)_\sigma}$ and $u_2=\sqrt{[\sigma^2\rho_s/\rho_n(\partial\sigma/\partial T)]}$ if we neglect the thermal expansion coefficient $\partial\rho/\partial T$, which is anomalously small for HeII and superfluid ^{3}He. In order to determine the coupling between first and second sounds, we assume that we excite first sound, i.e., pressure oscillations. From Eq. (5) we get

$$T'=P'\frac{\frac{\partial\sigma}{\partial P}u_1^2}{\frac{\partial\sigma}{\partial T}(u_2^2-u_1^2)}=\alpha P', \tag{6}$$

so pressure oscillations are accompanied by temperature oscillations with an amplitude $\alpha P'$. Using the relations

$$\left(\frac{\partial\sigma}{\partial P}\right)_T=-\left(\frac{\partial V}{\partial T}\right)_P=\frac{1}{\rho^2}\left(\frac{\partial\rho}{\partial T}\right)_P, \tag{7}$$

the amplitude T' is vanishingly small by virtue of the anomalously small magnitude of $\partial\rho/\partial T$.

Now consider the excitation of temperature oscillations and the propagation of second sound. We get from Eq. (4)

$$P'=\frac{\partial\rho}{\partial T}\frac{u_1^2u_2^2}{u_1^2-u_2^2}T'=\beta T'. \tag{8}$$

P' is small because of the size of $\partial\rho/\partial T$; this weak coupling renders the observation of the sound conversion difficult in pure superfluids.

II. SOUND CONVERSION IN IMPURE SUPERFLUIDS

We will consider dilute mixtures of different impurities in superfluids.[10] ^{3}He-^{4}He mixtures can be considered as an example of an impure superfluid. Since impurities participate only in normal fluid flow we should add the continuity equation for impurities[1]

$$\frac{\partial}{\partial t}\rho c+\nabla\rho c\mathbf{v}_n=0 \tag{9}$$

to the system of hydrodynamic equations, including in it the additional variable $c=N_im_i/(N_im_i+N_sm_s)$, and rewrite the thermodynamic equality as

$$dE=Tds+\mu d\rho+Zdc+(\mathbf{v_n}-\mathbf{v_s},\mathbf{dj}). \tag{10}$$

Here $Z=\rho(\mu_i-\mu_s)$, where μ_i is the chemical potential of the impurity and μ_s is that of the superfluid.

From the linearized system of hydrodynamic equations after substituting for the velocities $\mathbf{v_n}$ and $\mathbf{v_s}$ we get

$$\ddot{\rho}=\Delta P,\quad \frac{\rho_n\ddot{\sigma}}{\rho_s\sigma}=\sigma\Delta T+c\Delta\frac{Z}{\rho},\quad \frac{\dot{c}}{c}=\frac{\dot{\sigma}}{\sigma}. \tag{11}$$

Selecting the independent variables P', T', and c' and considering a traveling wave ($f\propto\exp[-i\omega(t-x/u)]$) we get:[1]

$$\left(\frac{\partial\rho}{\partial P}u^2-1\right)P'+\frac{\partial\rho}{\partial T}u^2T'+\frac{\partial\rho}{\partial c}u^2c'=0 \tag{12}$$

$$\left(\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial P}u^2-c\frac{\partial}{\partial P}\frac{Z}{\rho}\right)P'+\left(\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial T}u^2-\sigma-c\frac{\partial}{\partial T}\frac{Z}{\rho}\right)T' +\left(\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial c}u^2-c\frac{\partial}{\partial c}\frac{Z}{\rho}\right)c'=0 \tag{13}$$

$$c\frac{\partial\sigma}{\partial P}P'+c\frac{\partial\sigma}{\partial T}T'+\left(c\frac{\partial\sigma}{\partial c}-\sigma\right)c'=0 \tag{14}$$

The secular equation for this system determines the velocities of first and second sounds found[1] in the limit $\partial\rho/\partial T=0$.

In order to investigate sound conversion, let us isolate c' from Eq. (14) and exclude it from Eqs. (12) and (13):

$$P'\left[\left(\frac{\partial\rho}{\partial P}+\tilde{c}\right)u^2-1\right]+T'\left(\frac{\partial\rho}{\partial T}+\tilde{c}\right)u^2=0 \tag{15}$$

$$P'\left[\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial P}u^2-\frac{c}{\rho^2}\frac{\partial\rho}{\partial c}+\frac{c\frac{\partial\sigma}{\partial P}A}{B}\right] +T'\left[\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial T}u^2-\sigma+c\frac{\partial\sigma}{\partial c}+\frac{c\frac{\partial\sigma}{\partial T}A}{B}\right]=0. \tag{16}$$

Here $\tilde{c}=c/B(\partial\rho/\partial c)(\partial\sigma/\partial T)$; $A=(\rho_n/\rho_s\sigma)(\partial\sigma/\partial c)u^2-c(\partial/\partial c)(Z/\rho)$; and $B=c(\partial\sigma/\partial c)-\sigma$.

Conversion of first sound into second sound. From Eq. (16) one gets $A_1=A(u=u_1)$.

$$T'=-P'\frac{\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial P}u_1^2-\frac{c}{\rho^2}\frac{\partial\rho}{\partial c}+c\frac{\partial\sigma}{\partial P}\frac{A_1}{B}}{\frac{\rho_n}{\rho_s\sigma}\frac{\partial\sigma}{\partial T}u_1^2-\sigma+c\frac{\partial\sigma}{\partial c}+c\frac{\partial\sigma}{\partial T}\frac{A_1}{B}}=\alpha P'. \tag{17}$$

The coefficient α determines the amplitude of the temperature oscillations which accompany the pressure oscillations in the plane wave. While the coupling between P' and T' oscillations exists at all temperatures, temperature oscillations can propagate only below T_c (T_λ for HeII) . The excitation of pressure oscillations in impure He produces propagating temperature oscillations only below T_c (T_λ). The parametric decay of first sound and Cerenkov emission for ^{3}He-^{4}He mixtures have been considered in Ref. 11.

Conversion of second sound into first sound. From Eq. (15) one gets

$$P'=-T'\frac{\left(\frac{\partial\rho}{\partial T}+\tilde{c}\right)u_2^2}{\left(\frac{\partial\rho}{\partial P}+\tilde{c}\right)u_2^2-1}=\left(\frac{\partial\rho}{\partial T}+\tilde{c}\right)\frac{u_1^2u_2^2}{u_1^2-u_2^2}T'=\beta T'. \tag{18}$$

The small parameter $\partial\rho/\partial T$ is changed by $(\partial\rho/\partial T)+\tilde{c}$, which is proportional to $\partial\rho/\partial c$ and c. The conversion of second sound into first sound in impure superfluids is dominated by the term $c(\partial\rho/\partial c)$ rather then $\partial\rho/\partial T$ in pure superfluid and thus could be more easily observed.

III. SOUND CONVERSION IN AEROGEL

To analyze the sound conversion phenomena in superfluid He (^{3}He and ^{4}He) in aerogel we will use the modified hydrodynamic equations introduced for this case by McKenna *et al.*,[8] which means that the normal fluid is locked to the aerogel, and both move together with a velocity $\mathbf{v}_n$:

$$\dot{\rho}+\nabla(\rho_n\mathbf{v_n}+\rho_s\mathbf{v_s})=0;\qquad \frac{\partial\rho\sigma}{\partial \mathrm{t}}+\nabla(\rho\sigma\mathbf{v_n})=0; \tag{19}$$

$$\dot{\mathbf{v}}_{\mathbf{s}}=-\frac{1}{\rho}\nabla P+\sigma\nabla T;\qquad \dot{\rho}_a+\nabla(\rho_a\mathbf{v}_n)=0 \tag{20}$$

$$\rho_{na}\dot{\mathbf{v}}_n=-\frac{\rho_n}{\rho}\nabla P-\nabla P_a-\rho_s\sigma\nabla T. \tag{21}$$

These differ from the bulk superfluid He equations by the replacement $\rho_n\rightarrow\rho_n+\rho_a=\rho_{na}$ on the left-hand side of Eq. (21) and the additional restoring force P_a due to the aerogel.[8] Performing the same calculations as above we arrive at the following equations for P' and T' (excluding P_a' from the system of three equations for P', T', and P_a')

$$P'\left[\frac{u^2}{u_2^2}-\frac{1}{\rho}\left(\rho_s+\frac{\rho_n^2}{\rho_{na}}\right)-\frac{\rho_a\rho_n^2}{\rho\rho_{na}^2\left(\frac{u^2}{u_a^2}-\frac{\rho_a}{\rho_n}\right)}\right]$$
$$+T'\left[\frac{\partial\rho}{\partial T}u^2+\sigma\frac{\rho_s\rho_a}{\rho_n}+\frac{\sigma\rho_s\rho_a\rho_n}{\rho_{na}^2\left(\frac{u^2}{u_a^2}-\frac{\rho_a}{\rho_n}\right)}\right]=0 \tag{22}$$

$$P'\left[\rho\frac{\partial\sigma}{\partial P}u^2+\sigma\frac{\rho_s\rho_a}{\rho\rho_{na}}-\frac{\sigma\rho_a\rho_n\rho_s}{\rho\rho_{na}^2\left(\frac{u^2}{u_a^2}-\frac{\rho_a}{\rho_n}\right)}\right]+T'$$
$$\times\left[-\frac{\sigma^2\rho_s(\rho+\rho_a)}{\rho_{na}}+\rho\frac{\partial\sigma}{\partial T}u^2+\frac{\sigma\rho_a\rho_s^2}{\rho_{na}^2\left(\frac{u^2}{u_2^2}-\frac{\rho_a}{\rho_n}\right)}\right]=0. \tag{23}$$

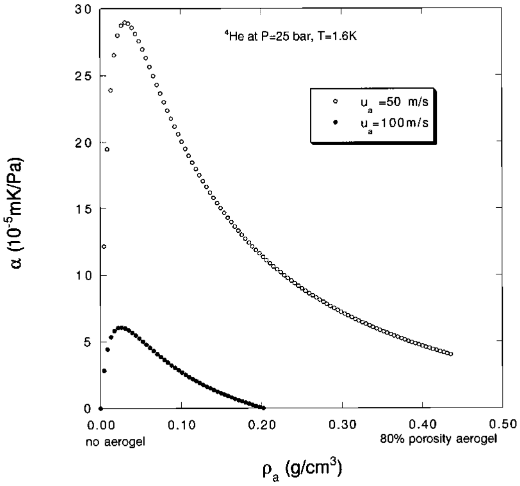

FIG. 1. The coupling parameter α of first sound to second sound versus aerogel density for superfluid ^{4}He.

Conversion of first sound into second sound. From Eq. (23) one obtains:

$$T'=-P'\frac{\rho\frac{\partial\sigma}{\partial P}u_1^2+\frac{\sigma\rho_a\rho_s}{\rho\rho_{na}}\left(1-\frac{\rho_n}{\rho_{na}A_1}\right)}{u_1^2\left(\rho\frac{\partial\sigma}{\partial T}+\frac{\sigma\rho_s}{\rho_{na}A_1}\frac{\partial\rho_a}{\partial T}\right)-\frac{\sigma^2\rho_s}{\rho_{na}}\left(\rho+\rho_a+\frac{\rho_a\rho_s}{\rho_{na}A_1}\right)}. \tag{24}$$

In the presence of aerogel, pressure waves produce temperature oscillations which could propagate only below T_c (T_λ).

Conversion of second sound into first sound. From Eq. (22) one gets

$$P'=-T'\frac{\left[u_2^2\left(\frac{\partial\rho}{\partial T}+\frac{\rho_n}{\rho_{na}A_2}\frac{\partial\rho_a}{\partial T}\right)+\sigma\frac{\rho_s\rho_a}{\rho_{na}}\left(1-\frac{\rho_n}{\rho_{na}A_2}\right)\right]}{\frac{u_2^2}{u_1^2}-\frac{1}{\rho}\left[\rho_s+\frac{\rho_n^2}{\rho_{na}}\left(1-\frac{\rho_a}{\rho_{na}A_2}\right)\right]}. \tag{25}$$

Here $A=(u^2/u_a^2)-(\rho_a/\rho_n)$, $A_i=A(u=u_i)$. Taking into account the small magnitude of $\partial\rho/\partial T$, $\partial\rho_a/\partial T$, $\partial\sigma/\partial P$, and also $u_2\ll u_1$ we simplify and rewrite Eqs. (24) and (25) as:

$$T'=-P'\frac{\frac{\sigma\rho_a\rho_s}{\rho\rho_{na}}\left(1-\frac{\rho_n}{\rho_{na}A_1}\right)}{u_1^2\rho\frac{\partial\sigma}{\partial T}},$$

$$P'=-T'\frac{\sigma\frac{\rho\rho_s\rho_a}{\rho_{na}}\left(1-\frac{\rho_n}{\rho_{na}A_2}\right)}{\rho_s+\frac{\rho_n^2}{\rho_{na}}\left(1-\frac{\rho_a}{\rho_{na}A_2}\right)}. \tag{26}$$

We plot the coefficient α for superfluid ^{4}He from Eq. (26) in Fig. 1.

Some conclusions follow from Eq. (26) and Fig. 1.

(i) It is much easier to observe sound conversion in superfluid ^{4}He in aerogel than in superfluid ^{3}He in aerogel. There is a three orders-of-magnitude difference of entropy density s in the superfluid regions of both isotopes (in K and mK regions). The use of denser aerogel in superfluid ^{4}He experiments[9] ($\rho_a = 0.05-0.25\ \mathrm{g/cm^3}$) also increases the observability of sound conversion phenomena in superfluid ^{4}He in aerogel in comparison with superfluid ^{3}He.

(ii) The coupling coefficients α and β have a maximum which depends on the values of ρ_s, ρ_a, and u_a.

(iii) The competition of the magnitudes of ρ_s and σ produces experimental constraints—an intermediate temperature regime is optimum for measurements: far from T_c where ρ_s is small and far from zero temperature, where entropy is equal to zero.

(iv) Via the large coefficient u_1^2 in denominator of the first of Eq. (26) the conversion of second sound into first sound in superfluid He in aerogel will be easier to observe than the reverse.

Estimates for the β coefficients for HeII and superfluid ^{3}He in aerogel at $T = 1.6$ K and $T = 0.7$ mK give the following results:

$$\beta_{\mathrm{HeII}} = 2.5\times 10^2\ \mathrm{Pa/mK}; \quad \beta_{^3\mathrm{He}} = 2\times 10^{-2}\ \mathrm{Pa/mK}, \tag{27}$$

while the coefficient α turns out to be a few orders of magnitude less (in mK/Pa). This is in agreement with the conclusion (4) above. For comparison in pure HeII $\beta = 30$ Pa/mK at $T = 1.3$ K,[4,6] which is about two orders of magnitude less.

IV. DOUBLE SOUND CONVERSION (DSC)

Some phenomena like the slow mode of superfluid ^{3}He in aerogel, observed by Golov *et al.*,[7] the slow mode in superfluid ^{4}He in aerogel[12] and heat pulse propagation with the velocity of the fast mode[9] can be understood as a double sound conversion (DSC) phenomena. When first sound is excited with a pressure amplitude P', the pressure oscillations below T_c are accompanied by temperature oscillations with an amplitude $T' = \alpha P'$ following from Eq. (18) or the first part Eq. (27). The temperature oscillations lead to pressure oscillations (below T_c) with amplitude P'', which is found from Eq. (19) or from the second part of Eq. (27) $P'' = \beta T' = \alpha\beta P'$. This is the DSC. In the sound experiments in superfluid ^{3}He in aerogel[7] a slow mode was observed. However, the slow mode was not excited by a direct thermal input. Instead a pressure oscillation (first sound) is converted into second sound below T_c, and the second sound is converted back to first sound which is detected by the transducer. A similar DSC phenomenon but of second sound into first sound and then back to second sound was reported by Mulders *et al.*[9] who observed the propagation of heat pulse in HeII in aerogel. The received pulses, detected by a bolometer, ‘‘*contained structure corresponding to the transit times of the fast mode.*’’[9] The origin of this observation was not understood; however, we can now attribute this behavior to the DSC phenomenon.

In conclusion we have put forward the framework for understanding sound conversion phenomena in different impure homogeneous superfluids. The observation of the sound conversion is enabled by impurities (including ^{3}He admixture in HeII) or by the presence of aerogel. We calculate the coupling between two sounds (first and second) for impure superfluids as well as for superfluid He (both ^{3}He and ^{4}He) in aerogel and show that coupling of sounds in these systems is provided either by $c(\partial\rho/\partial c)$ or by $\sigma\rho_a\rho_s$ (instead of $\partial\rho/\partial T$ which is enormously small in pure He) and thus could be more easily observed experimentally. This replacement plays a fundamental role in sound conversion phenomena in impure superfluids. It changes drastically everything (from the coupling strength of two sounds up to threshold values) for all three nonlinear processes: parametric decay of first sound, Cerenkov emission of second sound[13] as well as transformation of second sound into first sound.[4] It changes also the temperature regions where the Cerenkov threshold comes before the decay threshold and vice versa.[14] Thus the Cerenkov emission which has never been observed in pure superfluids should be observable in impure superfluids.

We would like to acknowledge helpful conversations with D. Lee, J. Reppy, A. J. Leggett, D. Osheroff, A. Golov, J. Hook, S. Garrett, G. Eska, V. Khmelenko, J. Beamish, N. Mulders, A. Matsubara, S. Kiselev, E. Smith, and E. Nazaretski. This research was supported by the NSF under Grant Nos. DMR-9705295 and DMR-9970817.

[1] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (W. A. Benjamin, New York, 1965).
[2] R. G. Arkhipov and I. M. Khalatnikov, Zh. Éksp. Teor. Fiz. **34**, 843 (1958) [Sov. Phys. JETP **7**, 583 (1958)].
[3] V. L. Pokrovsky and I. M. Khalatnikov, Zh. Éksp. Teor. Fiz. **71**, 1974 (1976) [Sov. Phys. JETP **44**, 1036 (1976)].
[4] S. Putterman and S. Garrett, J. Low Temp. Phys. **27**, 543 (1977).
[5] D. Rinberg, V. Cherepanov, and V. Steinberg, Phys. Rev. Lett. **76**, 2105 (1996).
[6] S. Garrett *et al.*, Phys. Rev. Lett. **41**, 413 (1978).
[7] A. Golov *et al.*, Phys. Rev. Lett. **82**, 3492 (1999).
[8] M. J. McKenna, T. Slawecki, and J. D. Maynard, Phys. Rev. Lett. **66**, 1878 (1991).
[9] N. Mulders *et al.*, Phys. Rev. Lett. **67**, 695 (1991).
[10] S. I. Kiselev, V. V. Khmelenko, and D. M. Lee, Fiz. Nizk. Temp. **26**, 874 (2000).
[11] V. Lebedev, Zh. Éksp. Teor. Fiz. **72**, 2224 (1977) [Sov. Phys. JETP **45**, 1169 (1977)].
[12] N. Mulders *et al.* (private communication).
[13] See, for example, Eqs. (3.2) and (3.5) of Ref. 3.
[14] See Fig. 3 and Eq. (4.4) of Ref. 3.