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ESTIMATION OF ACOUSTIC LOSSES IN THE QUALITY FACTOR OF A MICROMECHANICAL 2D RESONATOR

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A semi-analytical study of the acoustic radiation losses associated with various transverse vibration modes of a micromechanical (MEMS) annular resonator is presented. The quality factor, Q, of such resonators is of interest in many applications and depends on structural geometry, fluid-structure interaction, and the device encapsulation. Resonators with at least one surface exposed to air can display significant losses through acoustic radiation even at μm dimensions. In this study, well established analytical techniques for modeling resonator vibration modes and fluid-structure interaction are used in conjunction with numerical and symbolic computation to establish a semi-analytical procedure for computing Q due to acoustic radiation losses, Q_{ac}, in any vibrational mode. The procedure includes calculation of the exact modeshape using computer algebra with an FEM based calculation as the starting point. This modeshape is then used for acoustic field calculations that lead to the computation of Q_{ac}. The computed values of Q_{ac} show a non-monotonic variation with the increasing modal frequencies. This non-monotonic variation in Q_{ac} is attributed to the radiation efficiency associated with various modes of the annular resonator. There exists an inverse relationship between the acoustic radiation efficiency and the Q_{ac} for various resonant frequencies. With the increase in the number of nodal circles and diameters, the radiation efficiency decreases and hence Q_{ac} increases. Results are compared for the lowest two modes of a solid circular resonator using exact mode shapes to those of Lamb's approximate mode shapes. Comparison to published experimental results validates the predictive utility of the proposed technique, especially for higher modes where acoustic radiation is the dominant constituent of Q. The hybrid analytical-computational procedure established here can be used in choosing appropriate mode shapes of the resonator for the desired Q.

1. Introduction

Development of micromechanical resonators have many advantages such as low weight, low power requirement [1] and can also be batch processed by the microfabrication techniques. Dynamic micromechanical resonators such as RF filters [2], mass sensing [3], etc., are generally designed to have high quality factor (Q) [4]. Therefore, understanding the dominant dissipation mechanism that controls the quality factor is important in order to design sensitive sensors. In this paper, we quantify the losses due to acoustic radiation in an annular membrane fixed at its edges. Such problem has been of significant interest in a wide range of systems such as underwater structure [5] to the vibration of nano circular drum resonator in contact with surrounding fluid [6]. Such fluid-structure interaction problem induces two important effects, namely, the added mass effect and the acoustic radiation effect. In this paper, we show the effect of the acoustic radiation losses on the quality factor in higher modes of the vibrating annular plate with fixed outer boundaries.

Initial studies of fluid loading on vibrating rigid disk was observed by Lord Rayleigh [7]. Subsequently, Lamb [5] investigated the fluid loading effect on the first two resonant modes of a circular membrane clamped along its outer edge. Additionally, the effect of acoustic related losses was also found corresponding to those two modes. However, the acoustic loss computation was based on the approximate mode shapes. Using the exact mode shape, Amabili *et al*. [8, 9] analyzed the added mass effect on the resonant frequency of the structure using Hankel Transform. Such fluid-structure interaction effects have recently been analyzed in the context of Micro and Nano Electro Mechanical Systems (MEMS/NEMS) structures to study the added mass effect. Our main objective in the present study is to compute acoustic losses and the associated quality factor in different modes of an annular micromechanical resonator clamped at its outer edge.

To describe the acoustic radiation losses in various oscillatory modes of the annular plate, the corresponding mode shapes of the structure are obtained. Subsequently, the effects of the surrounding fluid on the associated natural frequencies and the Q-factor are analyzed using the corresponding mode shape. In the present study, the "added mass" effect on the frequencies of the structure is negligible. However, the surrounding air affects the Q-factor significantly because of the acoustic losses. Following the analytical approach proposed by Amabili *et al*. [8, 9], we compute the acoustic radiation loss based quality factor, Q_{ac}. Subsequently, we do comparisons, first, with Lamb's results for the first two modes, and then other experimental results [6] for the higher modes of the resonator. We observe a non-monotonic variation in Q_{ac} which may be due to the radiation efficiency associated with different modes of the annular resonator. Finally, we show that there exists an inverse relationship between the acoustic radiation efficiency, and the Q_{ac} for different resonant frequencies.

2. Mathematical Modeling

Here, we are interested in the acoustic radiation losses from an annular micromechanical resonator. Therefore, we consider an annular plate of inner radius b and outer radius a, with the outer edge fixed along the periphery and the inner edge completely free as shown in Fig. 1. For acoustic radiation, we only consider the upper hemispherical region of the fluid surrounding the structure, since the resonator is on a rigid substrate with a very small cavity between the resonator and the substrate.

2.1 Modal Analysis

First, we need to find the exact mode shape of the annular plate. Such analysis is not new; vibration of annular plates fixed at the outer boundary has been studied extensively and reported in the literature. We only need to capture the essential elements of the analysis here. Due to circular geometry, we use polar co-ordinates to facilitate computation. Using the standard thin plate theory with the assumptions of linear elastic, homogeneous and isotropic material, the governing equation

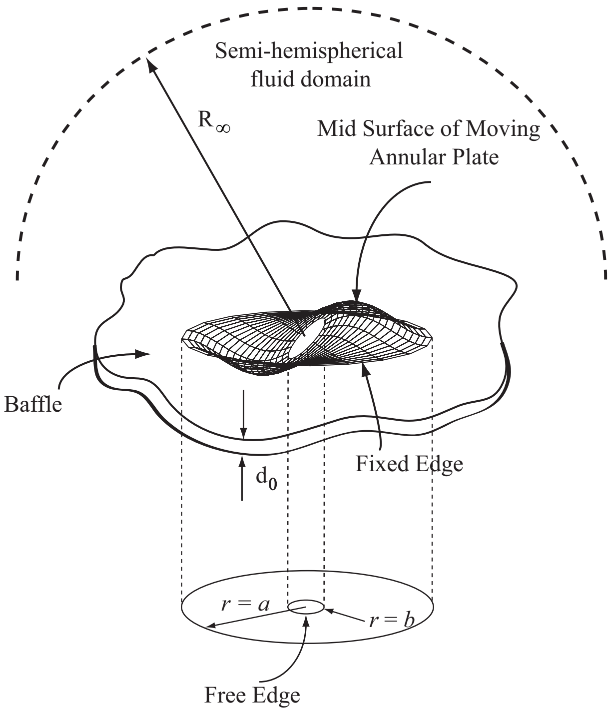


Figure 1. Schematic of semi-hemispherical fluid domain above the vibrating annular plate.

for the transverse vibration of the plate is, [10]

$$\nabla^4 W + \left(\frac{\rho_s d_0}{D}\right)\frac{\partial^2 W}{\partial t^2} = 0, \tag{1}$$

where, the Laplacian operator, $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$, and $D = \frac{E d_0^3}{12(1-\nu^2)}$ is flexural rigidity of the plate, ρ_s is the density, and d_0 is the uniform thickness of the plate. The modal solution to Eq. (1) with $W(r,\theta,t) = w(r,\theta)e^{i\omega t}$, is given in terms of Bessel's functions [11],

$$w(r,\theta) = [A_{mn}J_n(\beta r) + B_{mn}Y_n(\beta r) + C_{mn}I_n(\beta r) + D_{mn}K_n(\beta r)]\cos(n\theta), \tag{2}$$

where, m is the number of nodal circles, and n is the number of nodal diameters. The frequency parameter, $\beta^4 = \frac{\rho_s d_0}{D}\omega^2$ is obtained by solving the frequency equation with the appropriate boundary conditions by suitable root finding technique with initial guess, obtained by performing FEM modal analysis of the plate. The constants, A_{mn}, B_{mn}, C_{mn}, and D_{mn} are defined based on orthonormalization condition. For the solid circular plate, we take the limiting case of $b \rightarrow 0$, which results in the following expression for the transverse displacement.

$$w(r,\theta) = [A_{mn}J_n(\beta r) + C_{mn}I_n(\beta r)]\cos(n\theta). \tag{3}$$

The equations discussed above give natural frequencies and the corresponding modeshapes of the plate in vacuum. In the presence of a surrounding fluid, there is some mass loading due to the fluid that changes the natural frequencies. The mode shapes however, can still be assumed to remain unaffected as is commonly assumed in fluid-structure interaction problems [8, 9]. Here, we do include changes in the natural frequencies of the plate due to the fluid (air) loading. Our central problem is to estimate the energy carried away by the acoustic waves propagating in the fluid medium due to the oscillating plate. When the plate oscillates, it sets up a velocity field in the surrounding fluid which is best analyzed using the associated velocity potential, ϕ. The velocity potential is related to the pressure and the particle velocity through the following expressions [12],

$$p = -\rho_f \frac{\partial \phi}{\partial t} \text{ and } v = \nabla\phi. \tag{4}$$

For small pressure perturbations in a medium, the velocity potential must satisfy the following governing equation [13],

$$\nabla^2 \phi - \frac{1}{c_s^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \tag{5}$$

where, c_s is the speed of acoustic waves in the fluid. In our problem, the semi-infinite fluid domain above the oscillating plate imposes Sommerfeld boundary conditions of vanishing velocity and its gradient at the infinite boundary (as $r \to \infty$). With this condition, the real part of the velocity potential is given by,

$$\phi_r = Re\left(\frac{A(t) i^n e^{ik(c_s t - R)}}{R}\right)\int_b^a J_n(krsin(\xi))w(r,\theta)rdr, \tag{6}$$

where, $A(t)$ is the time dependent amplitude of oscillation, R is the radial distance from the center of the resonator to the acoustic field point at which the velocity potential is desired, θ is the azimuthal angle measured on the surface of plate. The power radiated by a particular mode across a hemispherical surface of radius R with its zenith angle variation ξ, is obtained by integration the acoustic intensity over the hemispherical fluid domain as, [5],

$$\frac{dE_{flux}}{dt} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} -\rho_f\left(\frac{\partial \phi_r}{\partial t}\frac{\partial \phi_r}{\partial R}R^2 sin(\xi)\right)d\xi d\theta. \tag{7}$$

The mean flux of energy emitted in the form of the sound waves is given by,

$$E_{\mathrm{MF}} = \frac{\int_0^{\frac{2\pi}{\omega}} \frac{dE_{\mathrm{flux}}}{dt}dt}{\int_0^{\frac{2\pi}{\omega}} dt} = UA^2(t). \tag{8}$$

The parameter U depends on fluid properties, geometric properties and frequency of oscillation of the resonator. We write the mean kinetic energy of the plate and adjacent fluid as, [5, 9]

$$E_{\mathrm{stored}} = T_P(1+\beta_{mn}) = VA^2(t), \tag{9}$$

where, T_P is the mean kinetic energy of the plate, and β_{mn} is the added virtual mass incremental factor [8, 9]. In steady state, the rate of addition of energy from the external source must be equal to the rate of energy emitted into the acoustic field. Therefore,

$$\frac{dA(t)}{dt} = -\delta A(t), \tag{10}$$

where, $\delta = U/V$, and is analogous to decay constant, $\zeta\omega$ [14]. Based on the definition of the quality factor [14], we get,

$$Q_{ac} = \frac{1}{2\zeta} = \frac{\omega}{2\delta\sqrt{1+\beta_{mn}}}. \tag{11}$$

The expression for Q_{ac} clearly shows that we need to find δ and β_{mn} in order to find Q_{ac}. But, both these quantities are mode dependent and their computation requires fair amount of algebra and numerics for evaluation of the involved derivatives and integrals. We accomplish this task by using MATLAB and MAPLE in tandem.

2.2 Acoustic Quality Factor for Approximate Mode Shapes

We now illustrate the computation of Q_{ac} given by Eq. (11) for the first two modes using Lamb's approximate mode shapes. Once the mode shape $w(r,\theta)$ is assumed or determined, we find the velocity potential ϕ given by Eq. (6) and proceed through Eqs. (7)-(10) to determine the equivalent decay constant, δ. Since δ is dependent on the mode shape, let us denote by a new variable α_i, where i is the mode number. We now present the expressions for the relevant quantities for computing Q_{ac} in the first two modes.

- Mode - 1: We use Lamb's approximate mode shape for mode-1, given by,

$$w(r) = C\left(1 - \frac{r^2}{a^2}\right)^2. \tag{12}$$

Note that this is an axisymmetric mode, and hence independent of θ. Using this mode shape, we find that,

$$\alpha_1 = \frac{5}{36}\frac{\rho_f}{\rho_s}\frac{\omega^2 a^2}{(1+\beta_{00})d_0 c_s}, \beta_{00} = 0.6689\frac{\rho_{air}}{\rho_s}\frac{a}{d_0}, \tag{13}$$

and therefore,

$$(Q_{ac})_i = \frac{1}{2\zeta} = \frac{\omega}{2\alpha_i\sqrt{1+\beta_{mn}}}. \tag{14}$$

- Mode - 2: The Lamb's approximate mode shape for diametral mode-2, is given by,

$$w(r,\theta) = C\left(1 - \frac{r^2}{a^2}\right)^2 r\cos(\theta). \tag{15}$$

The corresponding α_i and β_{mn} are given below,

$$\alpha_2 = \frac{5}{576}(1+\beta_{01})^{-1}\frac{\rho_f}{\rho_s}\frac{K^4 a^4 c_s}{(1+\beta_{01})d_0}, K = \frac{\omega}{c_s}, \text{ and } \beta_{01} = 0.3087\frac{\rho_{air}}{\rho_s}\frac{a}{d_0}. \tag{16}$$

The parameters computed for the approximate mode shapes are listed in Table 1.

Table 1. Approximate circular and diametral mode shapes associated Q-factors of circular solid plate.

Mode no.	(m, n)	$w(r,\theta)$	α_i	β_{mn}	$(Q_{ac})_i = \frac{\omega}{2\alpha_i\sqrt{1+\beta_{mn}}}$
1	(0, 0)	$C\left(1-\frac{r^2}{a^2}\right)^2$	1.064135e5	0.021129	99.9404
2	(0, 1)	$C\left(1-\frac{r^2}{a^2}\right)^2 r\cos(\theta)$	1.656637e5	0.009751	134.3513

3. Results and Discussions

To do the analysis, we take the following dimensions and material properties of the structure and the fluid domains based on the test structure used by Southworth *et al.* [6] in their experimental studies. Here, the annular plate is made of polysilicon with Young's modulus, $E = 150$ GPa, the density, $\rho_s = 2330\ \mathrm{kg/m^3}$, and the Poisson's ratio, $\nu = 0.22$. It has an outer radius of $a = 18.4\ \mu$m, inner radius of $b = 2\ \mu$m and a thickness of $d_0 = 300$ nm. The plate is surrounded by air having density, $\rho_f = 1.2\ \mathrm{kg/m^3}$ under constant ambient temperature, $T = 293$ K and pressure, $P = 1.013 \times 10^5$ Pa. For the analysis of the solid plate, we simply take the inner radius to be zero, i.e., b = 0 and rest of the parameters remain the same.

3.1 Solid Circular Plate

We first compare the results for the frequency and the quality factor in the first two modes of the resonator using (a) the exact mode shapes as given by Eq. (3), and (b) the approximate mode shapes proposed and used by Lamb [5]. We plot the normalized mode shapes, both exact and approximate, in Fig. 2 for the first two modes (m = 0, n = 0; and m = 0, n = 1). The difference in the amplitude along the spatial coordinates between the exact and the approximate mode shapes is clearly visible in the Figure. The quality factors computed for circular modes using ems and ams are 116 and 100 respectively, whereas for the diametric modes the corresponding quality factors are found to be 233

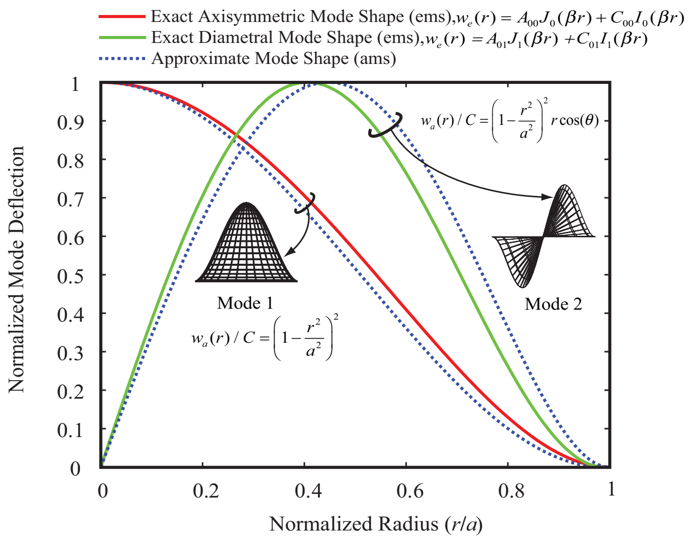


Figure 2. Normalized circular and diametral mode shape of a clamped solid circular plate.

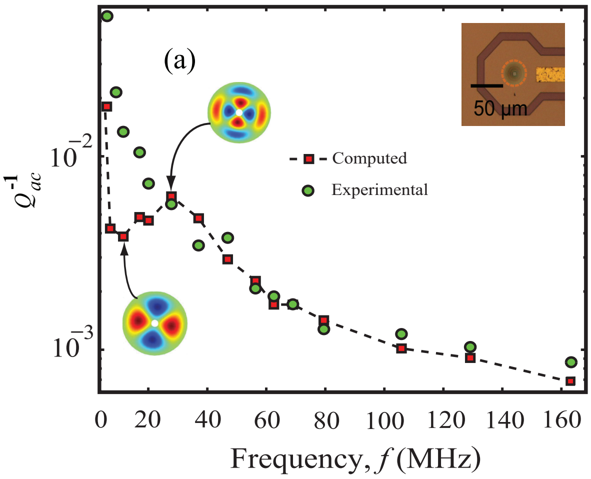
and 135. The difference in the computed Quality factors are non-negligible. The small amplitude variations between the exact mode shape and the approximate mode shape are sufficient to cause significant differences in the quality factors. In particular, the approximate mode shape of Lamb [5] for the second mode results in an over estimation of the radiation losses by as much as 42%.

3.2 Annular Plate

Our mathematical formulation allows us to analyze the annular plate withe equal ease. We consider the annular plate shown in Fig. 1 and find the acoustic losses and the corresponding quality factors for several mode shapes. We compare our results with the experimental results of Southworth *et al.* [6]. Therefore, we take the geometry and the material properties used in Southworth *et al.* for their drum resonator. For this resonator, the annular plate has fixed outer edge with the ratio of inner to outer radii as 0.1. Moreover, the lower surface of the plate is close to another fixed-fixed plate but the upper surface is open to the surrounding air. Based on this geometric configuration, there are two major source of energy dissipation: the squeeze film damping due to the trapped air film in the cavity below the resonator and the acoustic damping due to the acoustic radiation in the free space above the resonator surface. We use the procedure discussed above in Section 2 to compute the corresponding Q for each mode. The results are shown in Fig. 3(a).

We also include the experimental results by Southworth *et al.* in the Fig. 3(a). We limit our attention here to the acoustic radiative losses and the corresponding quality factor Q_{ac} in different modes of vibration. Here, acoustic radiation is not the only damping mechanism; we do not expect to match the experimentally obtained Q values with the computed Q_{ac}. A comparison of the two Q's in the Fig. 3(a) clearly indicates the relative contribution of Q_{ac} is very low in the lower modes of vibration and becomes dominant in the higher modes. It is well known that squeeze film damping is high at lower frequencies [15] and hence probably contributes maximum to the damping up to the fourth mode of vibration. It is interesting to see that the variation of Q_{ac} is not monotonic with increasing frequency over the entire frequency range. It has a markedly different behavior in the lower modes, but settles down to a more predictable, almost monotonic trend, in the higher modes. The unsettled behavior in the lower modes can be explained qualitatively by looking at the corresponding mode shapes in Fig. 3(a) and considering the relative efficiency of these modes in transferring the resonator's energy into the acoustic radiative field.

In acoustics, this efficiency is captured by a term called acoustic radiation efficiency, σ [16] which is defined as the ratio of acoustic radiated power of a particular mode of annular resonator to the acoustic power radiated by a uniformly vibrating baffled piston at a frequency for which the piston



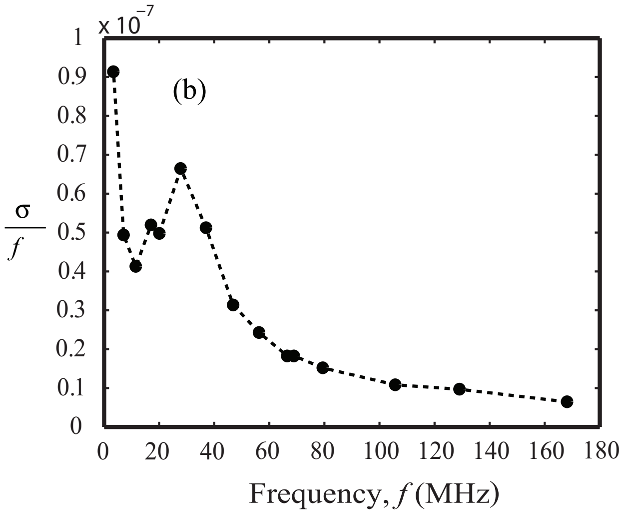


Figure 3. (a) Acoustic quality factor computed for experimental measurement carried out on a circular region as shown in SEM picture; (b) Acoustic radiation efficiency as a function of modal frequencies.

circumference greatly exceeds the acoustic wavelength. With some algebraic manipulations, one can show that, for a given mode shape, σ is related to Q_{ac} by the following expression,

$$\sigma = \frac{\rho_s d_0 (1+\beta_{mn})\omega}{\rho_f c_s Q_{ac}\sqrt{(1+\beta_{mn})}}. \tag{17}$$

We can compute σ for any given mode just from power computation and relate it to Q_{ac}. With increasing mode number, the nodal diameters and nodal circles increase in number, dividing the structure into several tiny vibrating substructures, effectively making it a collection of many point sources. After a certain number of such sources, addition of more sources makes less and less difference to Q_{ac} resulting in a monotonic and slow increase in Q_{ac}. Therefore, one can simply compute σ for a given resonator for various modes, normalize with frequency, and select the mode with the least σ, for the highest Q_{ac}. For the resonator considered here, there is considerable variation in σ at lower modes (see Fig. 3(b)). Based on σ, it is clearly found that the maximum power is radiated by the mode at approximately 27 MHz while the nearby local minimum exists at around 17 MHz radiating comparatively less power. However, as the number of nodal circles (m) and nodal diameters (n) increases, the mode shape start resembling an increasing number of point sources with decreasing radiation efficiency. It is largely because of larger areas associated with nodal regions which do not participate significantly in the motion, leading to monotonic but slow increase in Q_{ac} in higher modes.

4. Conclusions

In this paper, the computational procedure for estimating the acoustic losses using exact mode shapes corresponding to various modes of vibration of an annular plate, fixed only at its outer edge is presented. The mode shapes of a solid circular plate, as a limiting case of the annular plate, with inner radius set to zero, are also computed. The computed Q factors for the solid circular plate corresponding to Lamb's approximate mode shapes and the exact mode shape are compared. We have shown that the difference in the exact mode shape and the approximate mode shape results in significant change in the Q-factor, in particular as much as 42% in the second mode. The published experimental values that report the net Q for various annular plate modes are in good agreement with the computed acoustic Q factor. It is shown that for fifth and higher modes of vibration of annular plate, the comparison of experimentally observed net Q is in close agreement with the computed Q_{ac}

(the dominant acoustic losses contribute to 80% of experimental values). This study, thus, show that in higher modes of oscillation of 2-D micro resonators, acoustic radiation losses dominate the net Q of the device. Therefore, one can select the mode corresponding to the least radiation efficiency in order to have a high Q operation of the resonator.

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