Supplemental Information for
Acoustic properties of amorphous silica between 1 and 500 mK

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A review of the standard tunneling model (STM) is given in [1]. Here we discuss the acoustic properties predicted by the STM, following [1]. Tunneling states in a glass are approximated by TLS with asymmetry Δ and tunneling amplitude Δ_0. It is typically assumed that the distribution of these parameters is given by

$$P(\Delta, \Delta_0) = P_0/\Delta_0. \tag{1}$$

The quantities of experimental interest are the internal friction Q^{-1} and $\delta v/v_0$, the change in sound velocity relative to the sound velocity of the host $v_0 \equiv v(T=0)$. Two processes contribute to Q^{-1} and $\delta v/v_0$: (1) resonant interactions between the tunneling states and the phonons and (2) phonon-driven relaxation of the ensemble of tunneling states.

Resonant processes contribute to Q^{-1} a term proportional to $\tanh(\hbar\omega/2k_BT)$, which is negligible because $\hbar\omega << 2k_BT$ in the present work. The resonant contribution to $\delta v/v_0$ can be obtained from the resonant contribution to Q^{-1} via the Kramers-Kronig relation, yielding

$$\left[\frac{\delta v}{v_0}\right]_{\rm res} = C\left[\mathrm{Re}\Psi\left(\frac{1}{2}+\frac{\hbar\omega}{2\pi\imath k_BT}\right) - \ln\frac{\hbar\omega}{k_BT}\right] \tag{2}$$

where Ψ is the digamma function and C is the tunneling strength (one of the fit parameters in the present work). The temperature dependence of the term containing Ψ is again negligible because $\hbar\omega << 2k_BT$.

Sound waves in the glass perturb the asymmetry Δ of the TLS, resulting in a non-equilibrium distribution of tunneling state occupancies and subsequent relaxation toward the equilibrium state. The relaxation process affects both the sound velocity and attenuation:

$$\left[\frac{\delta v}{v_0}\right]_{\rm rel} = -\frac{1}{2}\mathrm{Re}[I] \tag{3}$$

$$Q^{-1} = \mathrm{Im}[I], \tag{4}$$

where

$$I = \frac{C}{P_0 k_BT}\left\langle \frac{1-r}{\cosh^2(E/2k_BT)}\frac{ir\gamma_{\rm m}}{\omega + ir\gamma_{\rm m}}\right\rangle, \tag{5}$$

$r = (\Delta_0/E)^2$, $E = \sqrt{\Delta^2+\Delta_0^2}$, and $\gamma_{\rm m}(E)$ is the maximum tunneling state relaxation rate for energy splitting E. The angular brackets $\langle ... \rangle$ indicate the ensemble average with respect to the parameter distribution $P(r,E) = P_0/2r\sqrt{1-r}$, where P_0 is a constant. We have changed the variables of the distribution function from Δ, Δ_0 to r, E for mathematical convenience.

The phonon-driven relaxation rate

$$\gamma_{\rm ph} = \frac{a}{k_B^3}\Delta_0^2 E\coth\left(\frac{E}{2k_BT}\right), \tag{6}$$

where $a = 3g^2k_B^3/2\pi\rho v_0^5\hbar^4$, g is the TLS deformation potential, and ρ is the mass density of the host. The maximum rate for a given energy splitting E occurs in the symmetric state with $\Delta_0 = E$ so that

$$\gamma_{\rm ph,m} = \frac{a}{k_B^3}E^3\coth\left(\frac{E}{2k_BT}\right). \tag{7}$$

The interaction-driven relaxation rate [2, 3] is given by $\gamma_{\rm tr,m} = bT$, where b is a constant. The total tunneling state relaxation rate is then given by $\gamma_{\rm m} = \gamma_{\rm tr,m} + \gamma_{\rm ph,m}$, which is inserted into Eq. 5. We calculated the ensemble average numerically. In [4], an analytical approximation of the ensemble average was obtained by replacing the splitting E with the thermal energy k_BT in $\gamma_{\rm ph,m}$. However, for the parameters considered here, such an approximation is inaccurate. The parameters a, b and C are the fitting parameters that appear in our paper.

The resonant and relaxational contributions to $\delta v/v_0$ are additive so that

$$\delta v/v_0 = [\delta v/v_0]_{\rm res} + [\delta v/v_0]_{\rm rel}. \tag{8}$$

The maximum in $\delta v/v_0$ and onset of the temperature dependence of Q^{-1} occurs at T_ω, at which $\omega = \gamma_{\rm ph,m}(E = k_BT)$.

In the last part of our paper, we introduce a fourth parameter μ that appears in the TLS distribution function $P_0(1-r)^{\mu-1/2}/2r$ [5], which reduces to that in Eq. 1 for $\mu = 0$. This distribution function is then used for the ensemble average of Eq. 5 and in the calculation that leads to Eq. 2. As a result, Eq. 2 is modified to

$$\left[\frac{\delta v}{v_0}\right]_{\rm res} = C\frac{\beta(1,\mu+0.5)}{2}\left[\mathrm{Re}\Psi\left(\frac{1}{2}+\frac{\hbar\omega}{2\pi\imath k_BT}\right) - \ln\frac{\hbar\omega}{k_BT}\right] \tag{9}$$

where β is the beta function defined as

$$\beta(a,b) = \int_0^1 dx x^{-1+a}(1-x)^{-1+b}. \tag{10}$$

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